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OFFICE OF NAVAL RESEARCH

Contract N7onr-35801

T. O. I.

NR-041-032

Technical Report No. 82

DYNAMIC TRANSVERSE LOADING OF BEAMS OF A MATERIAL EXHIBITING LINEAR STRAIN-HARDENING

by

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PROVIDENCE, R. I.

November, 1952

A11-82/24

DYNAMIC TRANSVERSE LOADING OF BEAMS OF A MATERIAL EXHIBITING LINEAR STRAIN-HARDENING[1]

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Abstract. The object of this report is to show that in certain circumstances the plastic deformation of beams, made of a material with linear strain-hardening and subject to dynamic transverse loading, can be determined by the techniques used in solving elastic problems. In particular, the differential equation of motion for such beams is in some instances of the same form as in the corresponding linear elastic case, and so any of the methods employed for solving elastic beam problems, such as the normal mode method, Laplace transform method, or Boussinesq's solutions for infinite beams, can be used. Because of this linear character of the differential equation of motion encountered in the analysis presented here, it is also shown that some initial motion problems for beams undergoing large plastic deformations due to transverse loading can be solved by superposing solutions. In these problems the disturbance part of the solution is obtained by some elasticity technique and is then superposed on the initial motion of the beam.

The method of solution is demonstrated by means of several examples involving finite beams. The first example is an initial motion problem and illustrates the method of superposition. The

1. The results presented in this paper were obtained in the course of research sponsored by the Office of Naval Research under Contract N7onr-35801 with Brown University.

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disturbance part of the solution is found by the normal mode technique, commonly used to solve forced vibration problems in elasticity. An approximation to the solution of the same problem is found by means of Boussinesq's solution for an infinite elastic beam. The second example is an initial stress problem for a simply supported beam. Again the method of superposition is used. The last example is a free boundary problem for a cantilever beam. This problem is solved by an inverse method whereby the form of the solution is assumed and the physical problem associated with this solution is then determined. (A similar type of analysis is used in finding solutions of elastic plane strain problems by the consideration of simple polynomial solutions of the biharmonic equation.)

The general problem of the determination of the plastic deformation of beams subject to dynamic transverse loading is very difficult to handle. Thus far only one such problem has been solved for a beam made of strain-hardening material. H. F. Bohnenblust [1]* has made an elastic-plastic analysis of the problem of an infinite beam subject to a constant velocity impact. In view, then, of the difficulty encountered in solving problems of this type, the analysis carried out in this report, while it applies only to a very special class of problems, seems well worthwhile, since it does add a group of tractable solutions to the literature. It is believed that this analysis represents the first treatment of the plastic deformation of finite beams, made of a strain-hardening material and subject to dynamic transverse loading.

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* Numbers in square brackets refer to the Bibliography at the end of the paper.

Nomenclature:

x Distance along the beam.

y Deflection.

K $\partial^2 y/\partial x^2$, the curvature of the deflection curve.

m Mass per unit length of the beam.

M Bending moment.

Q Shearing force.

t Time.

M_o The limiting value of M for rigid body motion.

Foot-pound-second units are used throughout this report.

1. Introduction and basic assumptions. The analysis carried out in this report is introduced to determine the permanent deformation of beams subjected to transverse loading of such a magnitude that the plastic strains produced are large compared to the elastic strains. It is shown here that, in some instances, a satisfactory approximation to the solution of such problems can be obtained by means of the same techniques used in elastic beam theory.

The analysis is based on the assumption of a linear strain-hardening bending moment-curvature relationship of the type shown in Fig. 1. Thus, elastic strains are ignored and the beam is considered to be either rigid or plastic. An approximation for the solution of the actual elastic-plastic problem is, then, obtained by neglecting the elastic strains and carrying out a plastic-rigid type of analysis. It is expected that this approximation will be satisfactory when the plastic strains are large compared to the elastic strains.

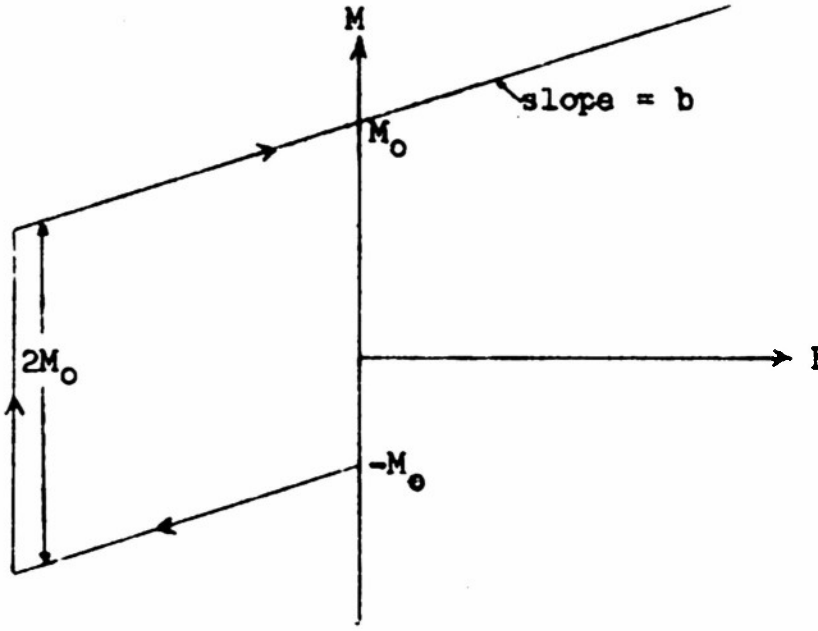

Fig. 1

The analysis is also based on the assumption that the rate of change of curvature of the beam is everywhere greater than or equal to zero, or everywhere less than or equal to zero, for the range of time considered, and this of course places certain restrictions on the initial conditions and boundary conditions for application of this analysis, as described below.

Under these assumptions the bending moment-curvature relationship of the beam for plastic flow is the linear relationship

$$M = \pm M_o + bK = \pm M_o + b \frac{\partial^2 y}{\partial x^2} \tag{1}$$

where the sign of M_o is the same as the sign of the rate of change of curvature of the beam. The differential equation of motion for the beam, which is obtained from the equilibrium conditions

$$Q = \frac{\partial M}{\partial x}, \qquad \frac{\partial Q}{\partial x} + m \frac{\partial^2 y}{\partial t^2} = 0,$$

becomes

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = 0 \qquad (2)$$

where $\frac{1}{c^2} = \frac{m}{b}$.

This equation of motion is of the same form as in the elastic beam theory, where the flexural rigidity has been replaced by b. Thus, under the assumptions stated, it is clear that the plastic deformation of a beam can be determined by the same methods used to solve the corresponding elastic problem and, since the equation of motion is linear, superposition of effects is permissible.

The assumptions stated here restrict the application of the analysis to a very special class of problems. However, as mentioned before, the general problem of the plastic deformation of beams subject to dynamic transverse loading has not yielded to theoretical analysis, hence the analysis given here represents definite progress.

2. Problem I -- An initial motion problem. For the application of this method to an initial motion problem, the initial motion of the beam must be such that the final rate of change of curvature of the beam after loading is everywhere greater than or equal to zero or everywhere less than or equal to zero. That is, the rate of change of curvature of the beam due to the initial motion must be such as to eliminate any oscillations in the sign of the rate of change of curvature due to the transverse loading. An example of such an initial motion is afforded by the following problem.

Let a cantilever beam of uniform cross section and length ℓ have one end ($x = 0$) fixed, and be subject to an initial motion

$$y = 2.25tx^2 - \frac{x^2}{30} \qquad (3)$$

which is maintained for all time by a bending moment, $M_o + b(4.5t - \frac{1}{15})$, applied at the free end of the beam. For $t > 0$ a bending moment $9bt$ is applied to the free end ($x = \ell$) of the beam. The problem is to determine the subsequent motion of the beam.

The initial motion and the applied bending moment given here were chosen rather arbitrarily. Any other choices would serve as well, provided they are such that the rate of change of curvature of the beam after loading is everywhere greater than or equal to zero (or everywhere less than or equal to zero).

Due to the prescribed initial motion, the initial rate of change of curvature of the beam is everywhere greater than zero. Thus, the bending moment-curvature relationship is

$$M_I = M_o + bK_I \qquad (4)$$

where K_I and M_I are the bending moment and curvature corresponding to the initial motion. If the final rate of change of curvature of the beam due to the initial motion plus the disturbance remains everywhere positive, the linear bending moment-curvature relationship of Eq. (1) remains valid and from Eqs. (1) and (4) it is clear that

$$M_D = bK_D$$

where K_D and M_D are the additional bending moment and curvature

corresponding to the disturbance.

The final deformation of the beam can be obtained by superposing the disturbance deformation onto the initial motion deformation. The disturbance deformation can be determined by any of the methods used in elastic beam theory.

A. Boussinesq Solution

The disturbance deformation due to the application of the bending moment at $x = \ell$ can be found from Boussinesq's solution for an infinite beam until such time as this solution gives an appreciable disturbance at the fixed end of the beam. While the solution does not satisfy the fixed end boundary conditions, it yields an approximation to the disturbance deformation until such time as the deflection and slope of the beam at x = 0 provided by this solution become significant.

The Boussinesq solution [2] of Eq. (2) for a semi infinite beam, initially ($t = -\infty$) straight and at rest, and subject to the conditions

$$\frac{\partial^3 y}{\partial x^3} = 0 , \qquad \frac{\partial^2 y}{\partial x^2} = F_1'(ct)$$

at $x = \ell$, is

$$y = \frac{2}{\sqrt{\pi}} \int_0^\infty F_1\left(ct - \frac{\xi^2}{2\alpha^2}\right) \cos \frac{\alpha^2}{2} \, d\alpha \tag{5}$$

where $\xi = \ell - x$ and $F_1'(ct) = \frac{d}{d(ct)} F_1(ct)$. Hence $F_1(ct) = \int_{-\infty}^{ct} F_1'(ct) d(ct)$.

Now, since the bending moment applied at the free end of the cantilever is

$$M_D(t) = \begin{cases} 0 & \text{for} \quad t \leq 0 \\ 9bt & \text{for} \quad t \geq 0 \end{cases}$$

and

$$M_D = bK_D$$

it is seen that

$$\frac{\partial^3 y_D}{\partial x^3} \equiv 0, \qquad \frac{\partial^2 y_D}{\partial x^2} = F_1'(ct) = \begin{cases} 0 & \text{for } ct \leq 0 \\ \frac{9}{c}(ct) & \text{for } ct \geq 0 \end{cases}$$

at $x = \ell$.

It follows from Eq. (5) that

$$y_D = \frac{9}{\sqrt{\pi}c} \int_{(\ell-x)/(2ct)^{\frac{1}{2}}}^{\infty} \left(ct - \frac{(\ell-x)^2}{2\alpha^2}\right)^2 \cos\frac{\alpha^2}{2}\, d\alpha\,, \tag{6}$$

$$\frac{\partial y_D}{\partial x} = \frac{18}{\sqrt{\pi}c} \int_{(\ell-x)/(2ct)^{\frac{1}{2}}}^{\infty} \left(ct - \frac{(\ell-x)^2}{2\alpha^2}\right) \frac{(\ell-x)}{\alpha^2} \cos\frac{\alpha^2}{2}\, d\alpha,$$

and

$$K_D = \frac{\partial^2 y_D}{\partial x^2} = \frac{18}{\sqrt{\pi}c} \int_{(\ell-x)/(2ct)^{\frac{1}{2}}}^{\infty} \left(ct - \frac{(\ell-x)^2}{2\alpha^2}\right)^2 \sin\frac{\alpha^2}{2}\, d\alpha. \tag{7}$$

From Eqs. (3) and (7) it is clear that the final rate of change of curvature of the beam due to the initial motion plus the disturbance motion is

$$\frac{dK}{dt} = 4.5 + 9\left\{1 - 2S[(\ell-x)^2/4ct]\right\}$$

where $S[(\ell-x)^2/4ct]$ is the Fresnel integral, $\frac{1}{\sqrt{2\pi}}\int_0^{(\ell-x)^2/4ct} \frac{\sin\eta}{\sqrt{\eta}}\,d\eta$.

Since $S[(\ell-x)^2/4ct] < .75$ for all values of the argument, it follows that

$$\frac{dK}{dt} > 0$$

everywhere along the beam for all time. Thus, Eq. (2) is everywhere valid and superposition of effects is allowable. Superposing Eqs. (3) and (6), we obtain

$$y = 2.25tx^2 - \frac{x^2}{30} + \frac{9}{\sqrt{\pi}c}\int_{(\ell-x)/(2ct)^{\frac{1}{2}}}^{\infty} \left(ct - \frac{(\ell-x)^2}{2\alpha^2}\right)^2 \cos\frac{\alpha^2}{2}\,d\alpha.$$

This Boussinesq solution was calculated [3] at $t = .01$ and $t = .02$ for the case when $\ell = 3$ and $e^2 = b/m = 907$. The results are shown in Table I. A plot of y_D vs. x for $t = .02$ is shown by the dotted curve in Fig. 2.

B. Normal mode method.

This problem may also be solved by finding y_D by the well known method of normal modes, and then superposing this solution onto the initial motion of the beam.

The disturbance deformation, y_D, is then assumed to be of the form

$$y_D = \sum_{n=1}^{\infty} X_n(x)\varphi_n(t) \tag{8}$$

where the $X_n(x)$ are the normal modes for the cantilever beam and the $\varphi_n(t)$ are the normal coordinates [4] of the system.

Table I

	t = .01			t = .02		
x	y_I	y_D	y	y_I	y_D	y
0	0	+.00004	+.00004	0	+.00102	+.00102
.15	-.00024	-.00005	-.00029	+.00026	+.00173	+.00199
.30	-.00098	-.00015	-.00113	+.00105	+.00223	+.00328
.45	-.00219	-.00019	-.00238	+.00236	+.00245	+.00481
.60	-.00390	-.00013	-.00403	+.00420	+.00228	+.00648
.75	-.00609	+.00005	-.00604	+.00656	+.00146	+.00802
.90	-.00878	+.00029	-.00849	+.00945	+.00006	+.00951
1.05	-.01194	+.00052	-.01142	+.01286	-.00192	+.01094
1.20	-.01560	+.00062	-.01498	+.01680	-.00438	+.01242
1.35	-.01974	+.00050	-.01924	+.02126	-.00712	+.01414
1.50	-.02438	+.00008	-.02430	+.02625	-.00988	+.01637
1.65	-.02949	-.00064	-.03013	+.03176	-.01230	+.01946
1.80	-.03510	-.00156	-.03666	+.03780	-.01399	+.02381
1.95	-.04119	-.00254	-.04373	+.04436	-.01448	+.02988
2.10	-.04778	-.00332	-.05110	+.05145	-.01330	+.03815
2.25	-.05484	-.00362	-.05846	+.05906	-.00996	+.04910
2.40	-.06240	-.00311	-.06551	+.06720	-.00397	+.06323
2.55	-.07044	-.00142	-.07186	+.07586	+.00509	+.08095
2.70	-.07898	+.00175	-.07723	+.08505	+.01763	+.10268
2.85	-.08799	+.00668	-.08131	+.09476	+.03394	+.12870
3.00	-.09750	+.01355	-.08395	+.10500	+.05421	+.15921

It can easily be shown by the usual method of finding normal modes* [5] that

$$X_n(x) = -\left\{(\cos k_n x - \cosh k_n x)(\cos k_n\ell + \cosh k_n\ell) + (\sin k_n x - \sinh k_n x)(\sin k_n\ell - \sinh k_n\ell)\right\}\Big/(m\ell)^{\frac{1}{2}} \sin k_n\ell \sinh k_n\ell \tag{9}$$

where the k_n are obtained from the roots of the equation

$$\cosh k_n\ell \cos k_n\ell = -1.$$

Lagrange's equations of motion are simply

$$\ddot{\varphi}_n + p_n^2\varphi_n = \Phi_n$$

in which Φ_n denotes the force corresponding to the coordinate φ_n, and $p_n^2 = c^2k_n^4$.

In order to obtain a force Φ_n, assume that a small increase $\delta\varphi_n$ is given to a coordinate φ_n. The work done by the external bending moment on the end of the beam is

$$\Phi_n\delta\varphi_n = 9bt\delta\left(\frac{\partial y_D}{\partial x}\right)_{x=\ell}$$

and so

$$\Phi_n = 9bt\left\{\frac{-2k_n(\sin k_n\ell \cosh k_n\ell + \sinh k_n\ell \cos k_n\ell)}{-(m\ell)^{\frac{1}{2}} \sin k_n\ell \sinh k_n\ell}\right\}.$$

Substituting Φ_n into the equations of motion and taking into consideration the homogeneous initial conditions on y_D, we find

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* When the principal modes have been normalized $m\int_0^\ell X_n^2 dx = 1$.

$$\varphi_n = \frac{9b}{p_n^2}\left(t - \frac{1}{p_n}\sin p_n t\right)\left\{\frac{2k_n(\sin k_n\ell \cosh k_n\ell + \sinh k_n\ell \cos k_n\ell)}{(m\ell)^{\frac{1}{2}} \sin k_n\ell \sinh k_n\ell}\right\}$$

Finally, substituting φ_n into Eq. (8) and superposing the initial motion yields the solution

$$y = \left\{2.25t - \frac{1}{30}\right\}x^2 + \sum_{n=1}^{\infty} \frac{9b}{p_n^2}\left(t - \frac{1}{p_n}\sin p_n t\right) \cdot$$

$$\cdot\left\{\frac{2k_n(\sin k_n\ell \cosh k_n\ell + \sinh k_n\ell \cos k_n\ell)}{(m\ell)^{\frac{1}{2}} \sin k_n\ell \sinh k_n\ell}\right\}X_n$$

where the X_n are given by Eq. (9). However, the convergence of this expression is improved by subtracting out the quasi static deflection, $\frac{9}{2b}tx^2$, due to the application of the bending moment to the end of the beam. Upon expanding this in terms of the normal modes, it becomes clear that y may be rewritten

$$y = 2.25tx^2 - \frac{x^2}{30} + \frac{9}{2b}tx^2 - \sum_{n=1}^{\infty} \frac{9b}{p_n^2} \sin p_n t \cdot$$

$$\cdot\left\{\frac{2k_n(\sin k_n\ell \cosh k_n\ell + \sinh k_n\ell \cos k_n\ell}{(m\ell)^{\frac{1}{2}} \sin k_n\ell \sinh k_n\ell}\right\}X_n .$$

This normal mode solution was also calculated at t = .01 and t = .02 for the case when $\ell = 3$ and $c^2 = 907$. Six terms of the series were used. The results of these calculations are shown in Table II, and plots of y_I vs. x and y vs. x for t = .02 are shown in Fig. 3. A comparison of the disturbance deformation obtained at t = .02 by the Boussinesq and normal mode methods is shown in Fig. 2.

Table II

	t = .01			t = .02		
x	y_I	y_D	y	y_I	y_D	y
0	0	0	0	0	0	0
.15	-.00024	-.00007	-.00031	+.00026	+.00029	+.00055
.30	-.00098	-.00018	-.00116	+.00105	+.00091	+.00196
.45	-.00219	-.00024	-.00243	+.00236	+.00149	+.00385
.60	-.00390	-.00017	-.00407	+.00420	+.00172	+.00592
.75	-.00609	+.00004	-.00605	+.00656	+.00135	+.00791
.90	-.00878	+.00031	-.00847	+.00945	+.00027	+.00972
1.05	-.01194	+.00055	-.01139	+.01286	-.00154	+.01132
1.20	-.01560	+.00064	-.01496	+.01680	-.00399	+.01281
1.35	-.01974	+.00049	-.01925	+.02126	-.00685	+.01441
1.50	-.02438	+.00006	-.02432	+.02625	-.00980	+.01645
1.65	-.02949	-.00065	-.03014	+.03176	-.01242	+.01934
1.80	-.03510	-.00157	-.03667	+.03780	-.01424	+.02356
1.95	-.04119	-.00253	-.04372	+.04436	-.01474	+.02962
2.10	-.04778	-.00331	-.05109	+.05145	-.01346	+.03799
2.25	-.05484	-.00362	-.05846	+.05906	-.00997	+.04909
2.40	-.06240	-.00311	-.06551	+.06720	-.00385	+.06335
2.55	-.07044	-.00144	-.07188	+.07586	+.00527	+.08113
2.70	-.07898	+.00173	-.07725	+.08505	+.01774	+.10279
2.85	-.08799	+.00667	-.08132	+.09476	+.03391	+.12867
3.00	-.09750	+.01357	-.08393	+.10500	+.05397	+.15897

3. Problem II --Simply supported beam. Consider a beam of uniform cross section simply supported at the ends $x = 0$ and $x = \ell$. Initially the beam is at rest and subject to an external bending moment, $M_I \leq - M_o$, at both ends. At time zero an external load distribution

$$q(x,t) = q_o t \sin \frac{\pi x}{\ell}$$

is applied to the beam and the resulting deformation is to be determined.

Initially the beam is entirely plastic. If the motion due to the applied load or disturbance is such that the rate of change of curvature of the beam is everywhere negative, for all t, the bending moment-curvature relationship of the beam is

$$M = - M_o + bK$$

and

$$M_D = bK_D$$

where M_D and K_D are the additional bending moment and curvature due to the applied external load. Again the deformation of the beam can be determined by finding the disturbance deformation by the normal mode method used in elastic beam theory, and then superposing this onto the initial deformation.

It is assumed in solving this problem that the rate of change of curvature of the beam is everywhere negative, for all t. Once the solution has been obtained, it is readily verified that this assumption is valid.

The initial deformation of the beam is

$$y_I = \frac{M_I + M_o}{2b} (\ell - x)x$$

and the disturbance deformation of the beam is assumed to be of the form

$$y_D = \sum_{n=1}^{\infty} X_n(x)\varphi_n(t) \tag{10}$$

where the $X_n(x)$ are the normal modes for a beam simply supported at $x = 0$ and $x = \ell$. These normal modes are readily found to be

$$X_n = (2/m\ell)^{\frac{1}{2}} \sin \frac{n\pi}{\ell} x . \tag{11}$$

Again we have as the equations of motion

$$\ddot{\varphi}_n + p_n^2 \varphi_n = \Phi_n \tag{12}$$

in which Φ_n is the force corresponding to the coordinate φ_n and $p_n^2 = c^2\pi^4 n^4/\ell^4$. If a small increase $\delta\varphi_n$ is given to a coordinate φ_n, the work done by the external load will be

$$\Phi_n \delta\varphi_n = \int_0^{\ell} q_o t \sin \frac{\pi x}{\ell} (2/m\ell)^{\frac{1}{2}} \sin \frac{n\pi}{\ell} x \delta\varphi_n) dx$$

and thus

$$\begin{cases} \Phi_n = 0 & \text{for } n \neq 1 \\ \Phi_1 = (2/m\ell)^{\frac{1}{2}}(\ell/2) q_o t . \end{cases}$$

Substituting these into the equations of motion and taking into consideration the initial conditions of the problem, we obtain

$$\begin{cases} \varphi_n = 0 & \text{for } n \neq 1 \\ \varphi_1 = (2/m\ell)^{\frac{1}{2}}(\ell/2) q_o \frac{1}{p_1^2} (t - \frac{1}{p_1} \sin p_1 t). \end{cases}$$

Finally, substituting X_n and φ_n into Eq. (10) and superposing the initial deformation, the following solution for the displace-

ments is obtained:

$$y(x,t) = \frac{M_I + M_O}{2b}(\ell - x)x + \frac{q_o}{mp_1^2} \sin \frac{\pi x}{\ell} \left\{ t - \frac{1}{p_1} \sin p_1 t \right\}.$$

To ensure that this is the solution of the problem, it remains only to verify that $\frac{dK}{dt} < 0$ for $0 \leq x \leq \ell$ and all $t > 0$. But, differentiating $y(x,t)$, we find

$$\frac{dK}{dt} = - \frac{q_o}{mp_1^2} \frac{\pi^2}{\ell^2} \sin \frac{\pi x}{\ell} \left\{ 1 - \cos p_1 t \right\}$$

which is readily seen to be negative for $0 \leq x \leq \ell$ and all $t > 0$.

4. Problem III -- A free boundary problem. In the two previous problems the beam was initially entirely plastic and remained entirely plastic for all time. No plastic-rigid boundaries were present. The absence of such boundaries accounts in large measure for the ease with which these problems were solved. However, in the case of a beam which is initially rigid, the application of a load will in general produce a moving plastic-rigid boundary. The determination of this free boundary and the solution of such problems does not appear simple. However, by the inverse method shown below, it is possible to find solutions to some free boundary problems under the assumptions of Sec. 1. The method consists essentially of determining what physical problems are associated with simple analytic expressions satisfying Eq. (2).

Consider the following solution of Eq. (2):

$$y = x^2t^2 - \frac{x^6}{180c^2} . \tag{13}$$

This expression satisfies the boundary conditions

$$\text{at } x = 0 \begin{cases} y = 0 \\ y' = 0 \end{cases} \quad \text{and} \quad \text{at } x = \xi(t) = (12)^{\frac{1}{4}}(ct)^{\frac{1}{2}} \begin{cases} K = \frac{\partial^2 y}{\partial x^2} = 0 \\ \frac{dK}{dt} = 0 \end{cases} .$$

That is, the above solution of the differential equation of motion for plastic flow satisfies fixed end conditions at $x = 0$ and free boundary conditions at $x = \xi(t)$. As t increases from $t = 0$, the position of the free boundary moves from the fixed end ($x = 0$) to larger values of x. Moreover, since $K = 0$ for $x = \xi(t)$, it appears that Eq. (13) may be the solution in the plastic region, $0 \leq x \leq \xi(t)$, for a cantilever problem in which the beam is initially straight. For $\xi(t) < x \leq \ell$ the beam remains rigid; ℓ is the length of the beam. This solution will be valid only until such time as the plastic-rigid boundary reaches the free end of the beam; that is, for $0 \leq t \leq \ell^2/c(12)^{\frac{1}{2}}$.

The physical problem associated with Eq. (13) is now assumed to be that of a cantilever beam, fixed at $x = 0$, initially straight and at rest, and subject at time zero to a force $F(t)$ and a bending moment $M(t)$, both of which are applied at the free end, $x = \ell$.

$F(t)$ and $M(t)$ are determined from the equations of motion for the rigid portion of the beam, i.e.,

$$\begin{cases} F(t) + b(y''')_{x=\xi} = \displaystyle\int_{\xi}^{\ell} (R - \xi)\ddot{\theta}\, m dR + \int_{\xi}^{\ell} \ddot{y} m dR \\[2ex] M(t) + F(t)(\ell - \xi) = M_0 + \displaystyle\int_{\xi}^{\ell} (R - \xi)\left\{(R - \xi)\ddot{\theta}\right\} m dR + \int_{\xi}^{\ell} (R - \xi)\ddot{y} m dR. \end{cases}$$

It follows, then, that

$$F(t) = - b(y''')_{x=\xi} + m(\ell - \xi)(\ddot{y})_{x=\xi} + m \frac{(\ell - \xi)^2}{2}(\ddot{\theta})_{x=\xi}$$

$$M(t) = - F(t)(\ell - \xi) + M_o + m \frac{(\ell - \xi)^3}{3}(\ddot{\theta})_{x=\xi} + m \frac{(\ell - \xi)^2}{2}(\ddot{y})_{x=\xi}$$

where

$$\xi(t) = (12)^{\frac{1}{4}}(ct)^{\frac{1}{2}}$$

$$(\ddot{y})_{x=\xi} = \left\{ \frac{\partial^2 y}{\partial t^2} + 2\dot{\xi} \frac{\partial^2 y}{\partial x \partial t} + (\dot{\xi})^2 \frac{\partial^2 y}{\partial x^2} \right\}_{x=\xi} = 6(12)^{\frac{1}{2}} ct$$

$$(\ddot{\theta})_{x=\xi} = \left\{ \frac{\partial^3 y}{\partial t^2 \partial x} + 2\dot{\xi} \frac{\partial^3 y}{\partial x^2 \partial t} + (\dot{\xi})^2 \frac{\partial^3 y}{\partial x^3} \right\}_{x=\xi} = 6(12)^{\frac{1}{4}}(ct)^{\frac{1}{2}}$$

$$(y''')_{x=\xi} = - \frac{2\xi^3}{3c^2} .$$

For any given set of parameters a check must be made to make sure that $M(x) < M_o$ for $\xi < x \leq \ell$. That is, it must be verified that the yield limit is not exceeded in that portion of the beam which is assumed to be rigid. M(x) is readily obtained from the following equilibrium equation:

$$M(x) = M(t) + F(t)(\ell - x) - \int_x^\ell (R - x)\left\{(R - \xi)\ddot{\theta}\right\} m dR - \int_x^\ell m\ddot{y}(R - x)dR .$$

As a specific example, the following practical values of the parameters were chosen: $\ell = 5$ ft., $M_o = 17$ ft.-lbs., m = 1.2 lbs./ft., $c^2 = 907$ ft.^{4}/sec.^{2} . M(x) was calculated for $\xi \leq x \leq \ell$ at t = 0, .005, .02, .05, .10, .20 secs. From the

results of these calculations, which are shown in Fig. 4, it appears that for this set of parameters $M(x) < M_0$ for $\xi < x \leq \ell$. It does not seem feasible to make a rigorous analytic check of this fact.

5. Conclusion. The examples treated in this report serve to illustrate the fact that, under certain conditions, the plastic deformation of beams subject to dynamic transverse loading can be determined by the same techniques used in solving elastic beam problems. This is due to the fact that, under the assumptions stated in Sec. 1, the equation of motion for a beam undergoing large plastic deformation is of the same linear form as the equation of motion for an elastic beam.

Problems I and II were readily solved, largely due to the fact that in each case the beam was initially entirely plastic and remained entirely plastic for all time. If the beam is initially rigid, an applied load will, in general, produce a moving plastic-rigid boundary. The presence of such free boundaries makes these problems more difficult to handle. The same difficulty arises, of course, in solving problems involving the unloading of external forces, rather than the loading type of problems treated here.

As shown by Problem III, it is possible to find solutions to some free boundary problems under the assumptions of Sec. 1. The method of solution is essentially an inverse process whereby a simple analytic expression satisfying the differential equation of motion for plastic flow is considered, and the physical prob-

lem associated with this solution is then determined. The general problem of a beam subject to prescribed external forces giving rise to free boundaries is not so readily solved.

It was hoped that a numerical technique based on Hartree's method [6] might be devised for locating these free plastic-rigid boundaries at any given time. While there was no free boundary to be determined, Hartree's method was first tried on Problem I of this report. However, it was found that, due to the presence of the fourth derivative in the equation of motion of the beam, the results of a finite difference technique to solve the ordinary differential equation resulting from Hartree's method were poor. In the case of a free boundary problem, the satisfaction of the boundary condition, $\frac{dK}{dt} = 0$, would introduce further high derivatives, and it was virtually impossible to retain sufficient accuracy using finite difference to complete such a problem.

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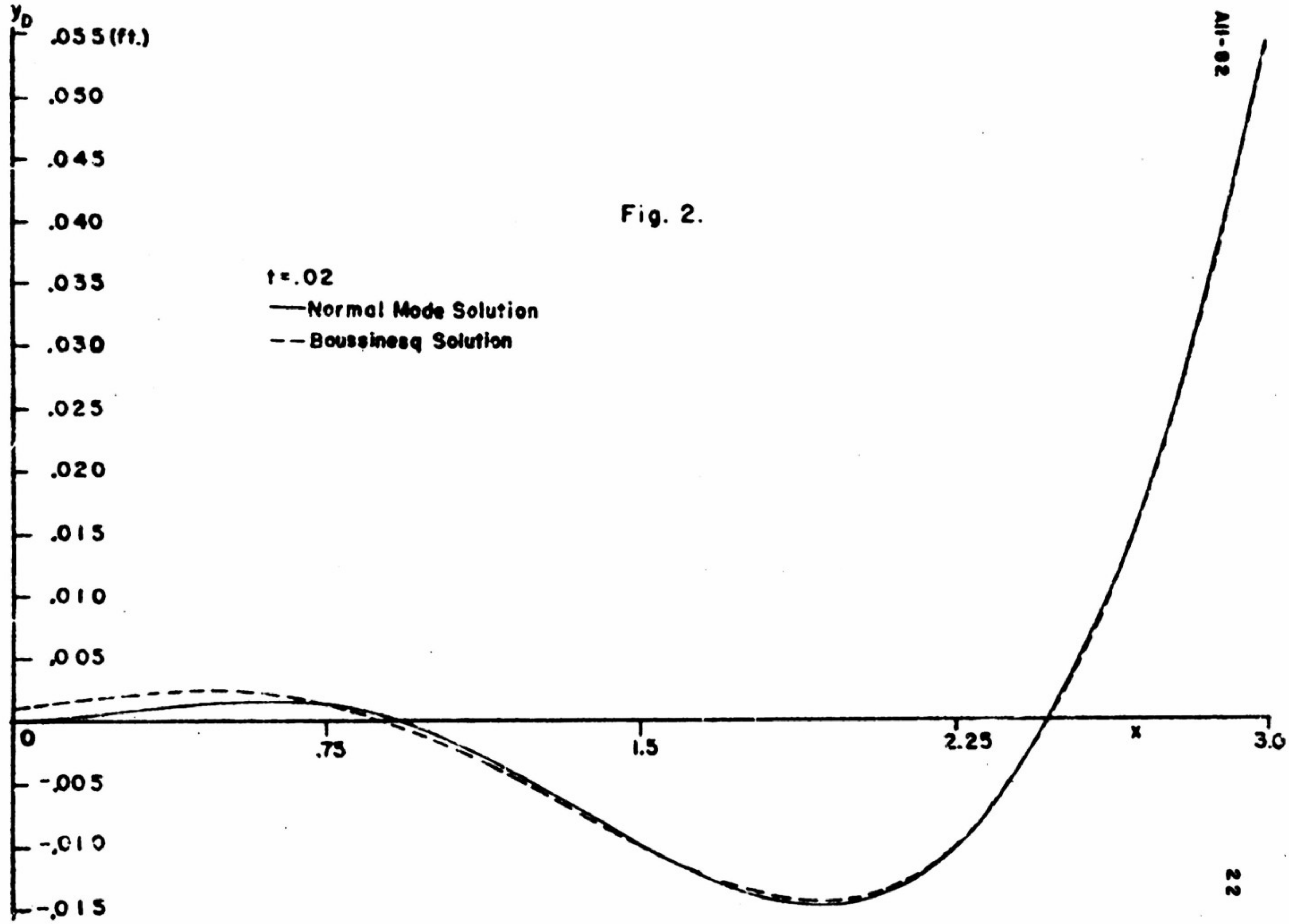

y_D
.055 (ft.)
.050
.045
.040
.035
.030
.025
.020
.015
.010
.005
0
-.005
-.010
-.015
.75
1.5
2.25
3.0
x
t = .02
——Normal Mode Solution
- - Boussinesq Solution

Fig. 2.

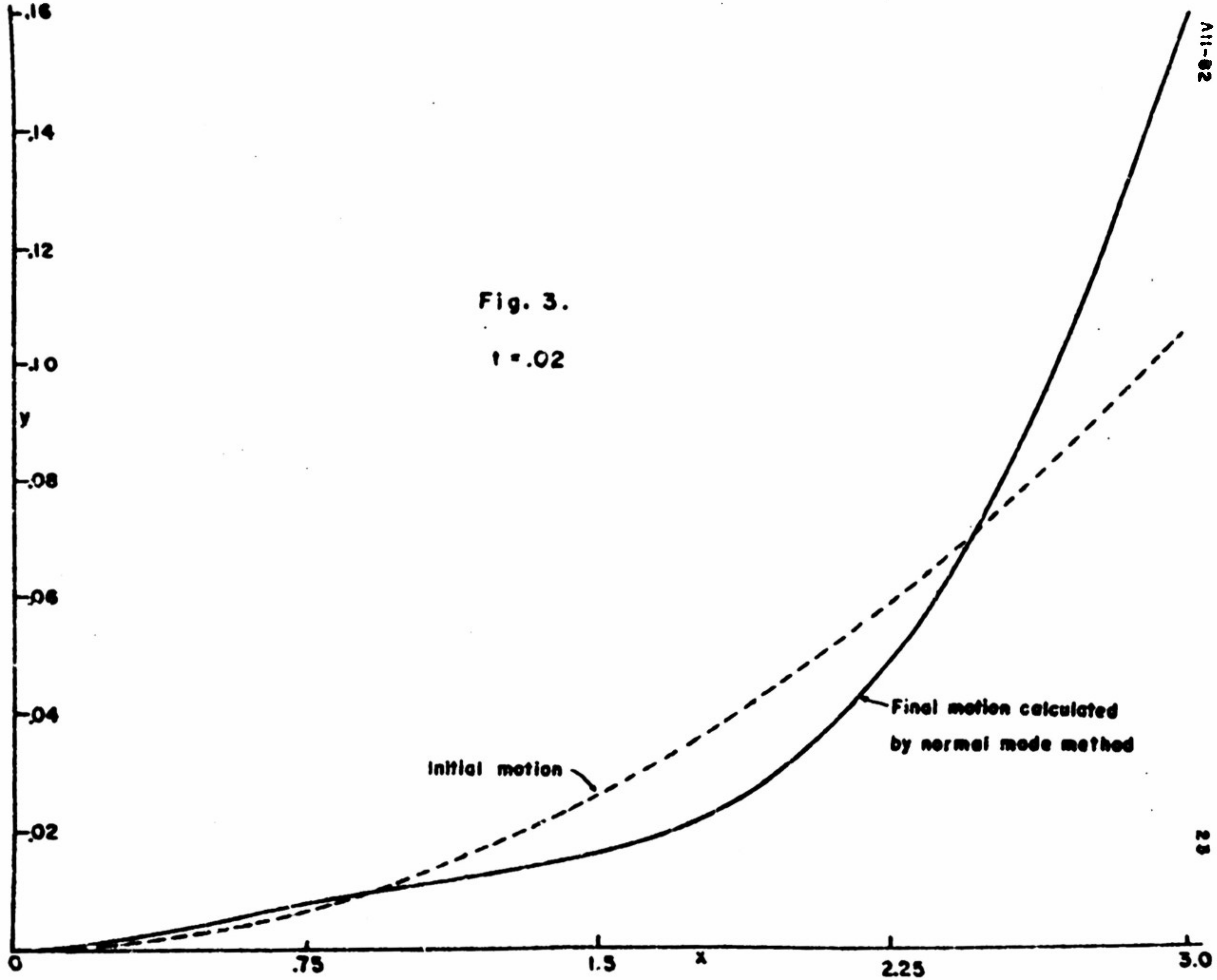
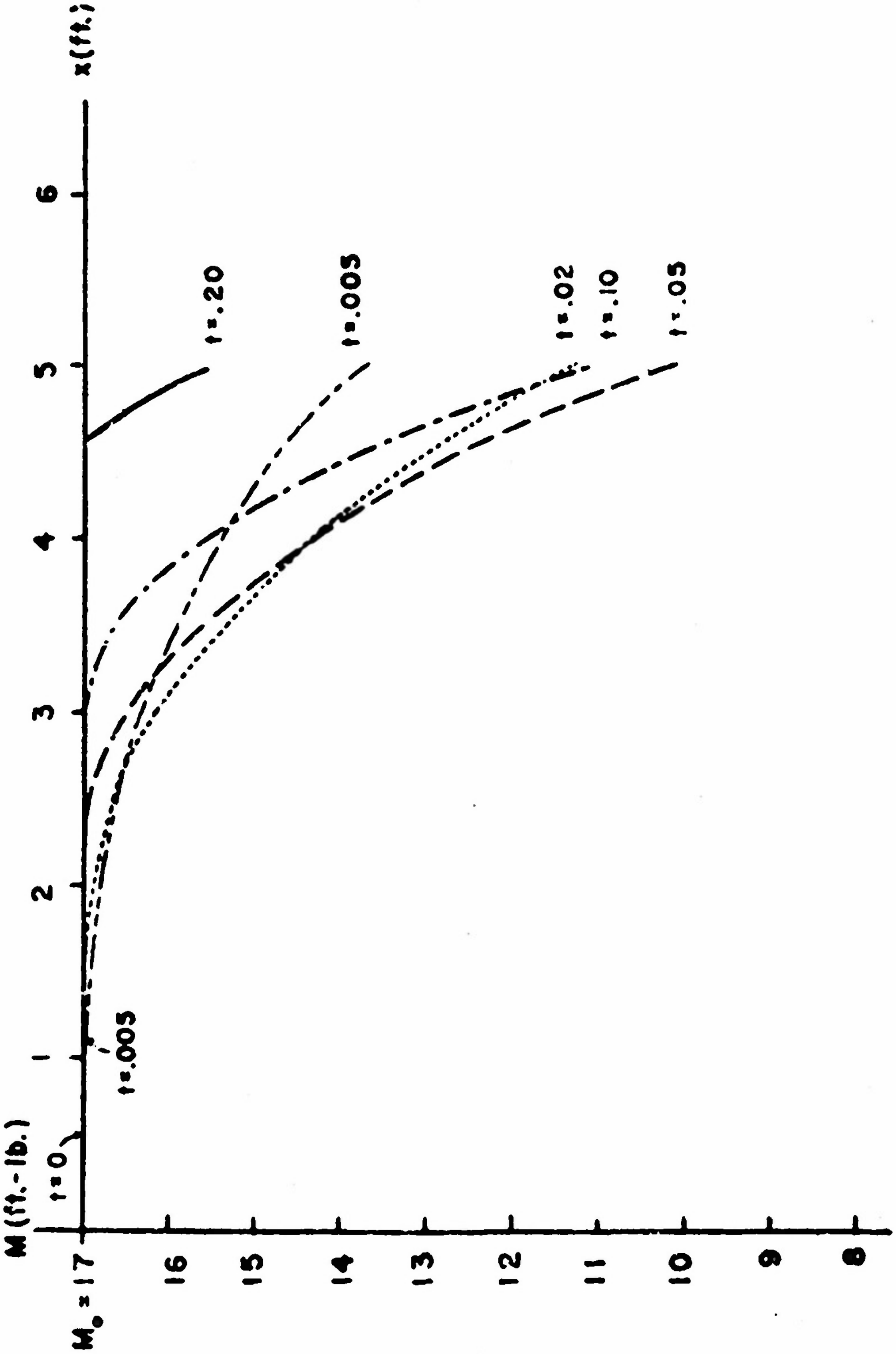

.16
.14
.12
.10
y
.08
.06
.04
.02
0
.75
1.5
x
2.25
3.0
Initial motion
Final motion calculated
by normal mode method

Fig. 3.

t = .02

Fig. 4.


x(ft.)
M(ft.-lb.)
M_o = 17
16
15
14
13
12
11
10
9
8
1
2
3
4
5
6
t=0
t=.005
t=.20
t=.005
t=.02
t=.10
t=.05